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A dynamic output feedback controllers for mismatched uncertain variable structure systems☆

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Abstract

This paper proposes a dynamic output feedback controllers for variable structure systems. It is based on the concept of output feedback controllers, introduced by Żak and Hui (IEEE Trans. Automat. Control AC-38 (1993) 1509) and Kwan (IEEE Trans. Automat. Control AC-41 (1996) 1691) for matched uncertain variable structure systems in which the state is unavailable and no estimated state is required. In this paper, we extend this idea to a class of mismatched uncertain variable structure systems. A modified variable structure controllers is derived to guarantee the existence of the sliding mode by using output feedback only. The stability of the equivalent reduced-order system in the sliding mode is assured under certain conditions.

Keywords: Output feedback; Variable structure systems; Mismatched uncertain systems

1. Introduction

The theory of variable structure control (VSC) is often used in controlling of uncertain systems. The main advantage of VSC exists in the sliding mode, where the system trajectories are constrained on the predetermined switching surfaces. In the sliding mode, if the uncertainties and/or disturbances of the system satisfy the so-called matching condition, the system behavior is insensitive to the matched internal parameter variations and external disturbances.

However, if the matching condition is not satisfied or the system suffers from mismatched uncertainties, then the system behavior in the sliding mode is not only governed by the switching surface but also determined by the mismatched uncertainties.

Another major drawback of VSC is that the state variables have to be accessible. In many practical systems, the state variables are not accessible for direct measurement or the number of measuring devices is limited. Thus, the design of asymptotic observers and dynamic compensators are very important and have been established (Bondaref, Bondaref, Kostyleva, & Utkin, 1985; Emelyanov, Korovin, Nersisyan, & Nisenzov, 1992; Diong & Medanic, 1992; Esfandiari & Khalil, 1992; Oh & Khalil, 1995). However, the direct output feedback design in variable structure systems (VSS) is worth investigating. Heck and Ferri (1989) proposed a direct output feedback in VSS by choosing a matrix such that the system satisfies the reaching condition. Żak and Hui (1993) proposed a static output feedback method. Some important conditions were given on the switching surface design. However, two major limitations of the paper Żak and Hui (1993) are difficult to achieve. The first is, the disturbances are bounded by a known function of outputs. The second limitation is the existence of a matrix equation ensuring the sliding condition. In Kwan (1996), a modified dynamic output feedback controller for a class of single-input/single-output (SISO) VSS was proposed. Under certain conditions, the applicability of output VSS can be greatly broadened. It should be pointed out that the above papers assume that

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the matching condition is satisfied. This is to say they only discuss the controllers design of matched uncertain VSS.

In this paper, we extend the idea of Żak and Hui (1993) and Kwan (1996) from matched uncertain VSS to a class of mismatched uncertain VSS. A modified controller using only output variable is proposed to stabilize the uncertain system robustly. Here the state is unavailable and no estimated state is required. We extend the design of Kwan (1996) from matched SISO systems to mismatched multi-input/multi-output (MIMO) systems. The major limitations of the method in Żak and Hui (1993) is also eliminated. Under certain conditions, the stability of the equivalent reduced-order system in the sliding mode is assured.

2. Statement of the problem

Consider the following mismatched uncertain systems:

$$\dot{x} = (A + \Delta A)x + B(u + \xi),$$
$$y = Cx, \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control input, $y \in R^p$ is the output. The term ΔA represents the mismatched uncertainty of the plant which the matching condition is not satisfied and ξ symbolizes the disturbances.

The sliding variable is defined as Żak and Hui (1993)

$$\sigma = Fy = FCx = Sx, \tag{2}$$

where $F \in R^{m \times p}$ is a constant matrix. The matrix F should be selected to satisfy

$$FC = S. \tag{3}$$

The design method to select F and S were given in Żak and Hui (1993). In the sliding mode, the desired distinct, non-zero, and real negative eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$ can be assigned.

There are two major assumptions in the paper Żak and Hui (1993). The first is the disturbances are bounded by a known function of outputs. That is $\|\xi(t)\| \leq \rho(t, y(t))$ with ρ a known function of outputs. This assumption is quite restrictive. The second assumption is that the condition $FCA = MC$ to guarantee sliding condition $\sigma = 0$. This condition is difficult to achieve.

With the above statements, the problems considered in this paper can be formulated as follows:

(a) Derive certain conditions which guarantee the mismatched uncertain system (1) in the sliding mode is asymptotically stable.

(b) Design a modified output feedback control which will eliminate the major limitations of the method by Żak and Hui (1993) and extend the method by Kwan (1996) from matched uncertain case to mismatched uncertain case.

(c) This controller guarantee; that the state trajectories of the mismatched uncertain system (1) can reach the switching surface $\sigma = 0$ in finite time and stay on it thereafter.

We assume the following to be valid:

Assumption 1. There exist known non-negative constants k_ξ and k_m such that $\|\xi\| \leq k_\xi + k_m\|x\|$.

Assumption 2. The pair (A, B) is controllable.

Assumption 3. The matrix S is existent (Żak & Hui, 1993, Theorem 4.1) and the equation $S = FC$ is solvable (Żak & Hui, 1993, Theorem 4.3).

The notation $\|\cdot\|$ in Assumptions 1 denotes the Euclidean norm of $(\cdot)$.

3. Stability in the sliding mode

In this section, some conditions are derived such that the system on the switching surface is stable even though the matching condition does not hold. First, the results obtained in El-Ghezawi et al. (1983) and Żak and Hui (1993), which are used to determining the switching surface. By Assumption 2, there exist matrices $W \in R^{n \times (n-m)}$ and $N \in R^{m \times n}$ such that $[A + BN]W = WJ$, where $J \in R^{(n-m) \times (n-m)}$ is a freely chosen Jordan matrix which determines the system dynamics restricted to the switching surface. The negative and real eigenvalues of J, λ_j, $j = 1, 2, \ldots, n-m$, are the desired eigenvalues in the sliding mode. Let $\lambda_{\max}(J)$ and $\lambda_{\min}(J)$ denote the maximum and minimum eigenvalues of J, respectively. Using procedures of El-Ghezawi et al. (1983) and Żak and Hui (1993), the following assumption is needed.

Assumption 4. The matrix $[W \ \ B]$ is invertible.

The inverse $[W \ \ B]$ has the form

$$\begin{bmatrix} W^{g} \\ B^{g} \end{bmatrix},$$

where W^g and B^g denote the generalized inverses of W and B, respectively. Selecting $S = B^g$ and a transformation matrix

$$\tilde{M} = \begin{bmatrix} W^{g} \\ S \end{bmatrix} \tag{4}$$

with $\tilde{M}^{-1} = [W\ B]$. Then system (1) can be written by

$$\dot{z} = W^{g}(A + \Delta A)Wz + W^{g}(A + \Delta A)B\sigma$$
$$\dot{\sigma} = S(A + \Delta A)Wz + S(A + \Delta A)B\sigma + u + \xi, \tag{5}$$

where $z = W^{g}x$ and $\sigma = Sx$.

The following theorem provides a condition that the mismatched uncertain system in the sliding mode is asymptotically stable.

Theorem 1. *Let* $\|\Delta A\| \le k_a$ *and*

$$P = \mathrm{diag}\{\lambda_{\min}(J)/\lambda_1, \lambda_{\min}(J)/\lambda_2, \ldots, \lambda_{\min}(J)/\lambda_{n-m}\}. \tag{6}$$

If

$$k_a < -\lambda_{\min}(J)/(\|PW^{g}\|\,\|W\|), \tag{7}$$

then the mismatched uncertain system (1) *in the sliding mode is asymptotically stable.*

Proof. In the sliding mode, we have $\sigma = 0$ and $\dot{\sigma} = 0$. Define a Lyapunov function candidate $V = z^{T}Pz$, where the positive-definite matrix P is defined in (6). If we differentiate V with respect to time combined with (5) and use the fact $W^{g}AW = J$, then

$$\dot{V} = z^{T}(J^{T}P + PJ)z + 2z^{T}PW^{g}\Delta AWz$$
$$\le z^{T}(J^{T}P + PJ)z + 2k_a\|z\|^2\|PW^{g}\|\,\|W\|.$$

It follows from (6) that

$$J^{T}P + PJ = \mathrm{diag}\{2\lambda_{\min}(J), 2\lambda_{\min}(J), \ldots, 2\lambda_{\min}(J)\}.$$

Then we obtain

$$\dot{V} \le 2\lambda_{\min}(J)\|z\|^2 + 2k_a\|z\|^2\|PW^{g}\|\,\|W\|.$$

So, if condition (7) is satisfied, the system in the sliding mode is asymptotically stable. □

Remark 1. In the sliding phase, the designer must select the minimum eigenvalue of J to satisfy the condition (7). If this (7) is not satisfied, one condition (Glazos & Żak, 1995, Theorem 1) can guarantee that the system in the sliding mode is globally uniformly practically stable.

4. The hitting phase design

Now, the modified variable structure controller is selected to be

$$u = -k_1\sigma - \{k_2\eta(t) + k_3\}\sigma/\|\sigma\| - \alpha\sigma/\|\sigma\|, \tag{8}$$

where $\alpha > 0$ and k_1, k_2 and k_3 are constant gains, $\eta(t)$ is a time function, and all will be designed later. It should be pointed out that the controller could use only the output signal. First, to design the function $\eta(t)$ for (8), we need the following lemma.

Lemma. *Assume* $C' \ge 0$, $r(t)$, $h(t)$ *and* $g(t)$ *are non-negative-valued continuous functions. If*

$$r(t) \le C' + \int_0^t h(\tau)r(\tau)\,d\tau + \int_0^t g(\tau)\,d\tau,$$

then

$$r(t) \le C'\exp\{f(t)\} + \int_0^t g(\tau)\exp\{f(t) - f(\tau)\}\,d\tau,$$

where

$$f(t) = \int_0^t h(\tau)\,d\tau.$$

Proof. Let

$$s(t) = C' + \int_0^t h(\tau)r(\tau)\,d\tau + \int_0^t g(\tau)\,d\tau,$$

then we have $r(t) \le s(t)$. Taking the time derivative of $s(t)$ yields

$$\dot{s}(t) = h(t)r(t) + g(t) \le h(t)s(t) + g(t),$$

then

$$\{\dot{s}(t) - h(t)s(t)\}\exp\{-f(t)\} \le g(t)\exp\{-f(t)\}.$$

Since $f(t) = \int_0^t h(\tau)\,d\tau$, we have

$$\frac{d}{dt}\{s(t)\exp\{-f(t)\}\} \le g(t)\exp\{-f(t)\}.$$

Integrating the above inequality on both sides, we obtain

$$s(t) \le C'\exp\{f(t)\} + \exp\{f(t)\}\int_0^t g(\tau)\exp\{-f(\tau)\}\,d\tau,$$

since $r(t) \le s(t)$, so we can show that

$$r(t) \le C'\exp\{f(t)\} + \int_0^t g(\tau)\exp\{f(t) - f(\tau)\}\,d\tau. \qquad \square$$

Remark 2. This lemma is a similar case of the Generalized Gronwall's Lemma (Hale, 1980).

Recall $\|\Delta A\| \le k_a$ and let

$$\|W^{g}\Delta AW\| \le k_a\|W^{g}\|\,\|W\| \equiv \rho_1$$
$$\|W^{g}\Delta AB\| \le k_a\|W^{g}\|\,\|B\| \equiv \rho_2. \tag{9}$$

Applying the Lemma, we derive the following theorem to get the function $\eta(t)$.

Theorem 2. *Consider the first equation of* (5)

$$\dot{z} = (J + W^{g}\Delta AW)z + (W^{g}AB + W^{g}\Delta AB)\sigma. \tag{10}$$

Let $\lambda_{\max}$ be the maximum eigenvalue of J. Then the following two statements hold:

(i) $\|\exp(Jt)\| \le k\exp(\lambda_{\max}t)$ *for some* $k > 0$.
(ii) $\|z\|$ *is bounded by* $\eta(t)$ *for all time, where* $\eta(t)$ *is the solution of*

$$\dot{\eta}(t) = \lambda\eta(t) + k(\|W^{\mathrm{g}}AB\| + \rho_2)\|\sigma\|, \quad \eta(0) \ge k\|z(0)\| \tag{11}$$

with

$$\lambda = \lambda_{\max} + k\rho_1 < 0.$$

Proof. Since all the eigenvalues of J are real negative, condition (i) holds obviously. To see (ii), we solve (10) to yield

$$\begin{aligned}\|z(t)\| &\le \|e^{Jt}\|\,\|z(0)\| + \int_0^t \|e^{J(t-\tau)}\|\,\|W^{\mathrm{g}}\Delta AWz(\tau)\\ &\qquad + (W^{\mathrm{g}}AB + W^{\mathrm{g}}\Delta AB)\sigma\|\,\mathrm{d}\tau\\ &\le k\exp(\lambda_{\max}t)\|z(0)\|\\ &\qquad + \int_0^t k\exp(\lambda_{\max}(t-\tau))\|W^{\mathrm{g}}\Delta AWz(\tau)\\ &\qquad + (W^{\mathrm{g}}AB + W^{\mathrm{g}}\Delta AB)\sigma\|\,\mathrm{d}\tau.\end{aligned}$$

For the above inequality, we multiply the term $\exp(-\lambda_{\max}t)$ to both sides and using (9), then

$$\begin{aligned}&\|z(t)\|\exp(-\lambda_{\max}t)\\ &\quad \le k\|z(0)\| + \int_0^t k\exp(-\lambda_{\max}\tau)\rho_1\|z(\tau)\|\,\mathrm{d}\tau\\ &\qquad + \int_0^t k\exp(-\lambda_{\max}\tau)(\|W^{\mathrm{g}}AB\| + \rho_2)\|\sigma\|\,\mathrm{d}\tau.\end{aligned}$$

Let

$$r(t) = \|z(t)\|\exp(-\lambda_{\max}t)$$

$$C' = k\|z(0)\|$$

$$h(t) = k\rho_1$$

$$g(t) = k\exp(-\lambda_{\max}t)(\|W^{\mathrm{g}}AB\| + \rho_2)\|\sigma\|$$

$$f(t) = \int_0^t h(\tau)\,\mathrm{d}\tau = k\rho_1 t.$$

Applying the Lemma, we obtain

$$\begin{aligned}&\|z(t)\|\exp(-\lambda_{\max}t)\\ &\quad \le k\|z(0)\|\exp(k\rho_1 t)\\ &\qquad + \int_0^t k\exp(-\lambda_{\max}\tau)(\|W^{\mathrm{g}}AB\| + \rho_2)\|\sigma\|\\ &\qquad \times\exp(k\rho_1 t - k\rho_1\tau)\,\mathrm{d}\tau.\end{aligned}$$

Shift the term $\exp(-\lambda_{\max}t)$ to the right-hand side for the above inequality, we have

$$\begin{aligned}\|z(t)\| &\le k\|z(0)\|\exp\{(\lambda_{\max} + k\rho_1)t\}\\ &\qquad + \int_0^t k\exp\{(\lambda_{\max} + k\rho_1)(t-\tau)\}\\ &\qquad \times(\|W^{\mathrm{g}}AB\| + \rho_2)\|\sigma\|\,\mathrm{d}\tau\\ &\le \eta(0)\exp\{(\lambda_{\max} + k\rho_1)t\}\\ &\qquad + \int_0^t k\exp\{(\lambda_{\max} + k\rho_1)(t-\tau)\}\\ &\qquad \times(\|W^{\mathrm{g}}AB\| + \rho_2)\|\sigma\|\,\mathrm{d}\tau\\ &= \eta(t), \quad \text{if } \eta(0) \ge k\|z\| > 0,\end{aligned}$$

where $\eta(t)$ satisfies (11). Hence, we can see that $\eta(t) \ge \|z(t)\|$ for all time, if $\eta(0)$ is sufficiently large. □

Remark 3. Condition (i) holds because J is a freely chosen Jordan matrix. Its eigenvalues are all negative and real. In condition (ii), we must select the maximum eigenvalues of J, $\lambda_{\max}$, to meet the condition $\lambda < 0$.

Remark 4. Examine the controller (8), (2) and (11), we can know that the controller (8) use only the output signal indeed.

Now let us discuss the reaching conditions in the following theorem.

Theorem 3. *The mismatched uncertain system* (1) *under the controller* (8) *reach the switching surface* $\sigma = 0$ *in finite time and stay on it thereafter if the constant gains satisfy the following conditions*:

$$\begin{aligned}&k_1 > \|SAB\| + k_{\mathrm{a}}\|S\|\,\|B\| + k_m\|B\|,\\ &k_2 > \|SAW\| + k_{\mathrm{a}}\|S\|\,\|W\| + k_m\|W\|,\\ &k_3 > k_\xi.\end{aligned} \tag{12}$$

Proof. Since $x = Wz + B\sigma$, using Theorem 2 we have $\|x\| \le \|W\|\eta(t) + \|B\|\,\|\sigma\|$. It follows from (5) and Assumption 1 that

$$\begin{aligned}\sigma^{\mathrm{T}}\dot{\sigma} &= \sigma^{\mathrm{T}}SAWz + \sigma^{\mathrm{T}}SAB\sigma + \sigma^{\mathrm{T}}S\Delta AWz\\ &\qquad + \sigma^{\mathrm{T}}S\Delta AB\sigma + \sigma^{\mathrm{T}}(u + \xi)\\ &\le \|SAB\|\,\|\sigma\|^2 + k_{\mathrm{a}}\|S\|\,\|B\|\,\|\sigma\|^2\\ &\qquad + \|SAW\|\,\|\sigma\|\,\|z\| + k_{\mathrm{a}}\|S\|\,\|W\|\,\|\sigma\|\,\|z\|\\ &\qquad + (k_\xi + k_m\|x\|)\|\sigma\| + \sigma^{\mathrm{T}}u.\end{aligned} \tag{13}$$

Then using the controller in (8) and (12), the inequalities (13) becomes $\sigma^{T}\dot{\sigma} \leq -\alpha\|\sigma\|$. Hence the state trajectory will reach the switching surface in finite time and stay on it thereafter. □

Remark 5. The condition $\sigma^{T}\dot{\sigma} \leq -\alpha\|\sigma\|$ is called reaching condition that guarantee the sliding mode. It is easy to verify that the sliding mode is generated in finite time $\|\sigma(0)\|/\alpha$. On the other hand, the paper by Heck and Ferri (1989) proposed output feedback in VSS such that the system satisfies another reaching condition $\sigma^{T}\dot{\sigma} \leq -1$. And the sliding mode is generated in finite time $\sigma^{T}(0)\sigma(0)$.

5. Conclusions

Some new results of the design of output feedback controller for a class of uncertain VSS has been developed in detail. This modified controller use only output variable and no estimated state is required. In the sliding mode, asymptotic stability of the mismatched uncertain VSS is assured under certain conditions. Using some results and a dynamic variable $\eta(t)$, this new controller extend the stabilization of the output feedback VSS from matched uncertain case to mismatched uncertain case.

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